Momentum, Heat, and Mass Transfer in Turbulent Non-Newtonian Boundary Layers

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The boundary-layer equations and a Blasius type of relationship between f and $N_{Re\ gen}$ are used to derive expressions for velocity distribution, local boundary-layer thickness, local shear stress, and total drag force for the turbulent boundary-layer flow of a power law non-Newtonian fluid across a flat plate at zero incidence. Relationships are derived for the velocity at the edge of the laminar sublayer and for the thickness of the laminar sublayer.

An analogy between heat and momentum transfer is then used to obtain expressions for local and mean values of the heat transfer coefficient in a turbulent thermal boundary layer for power law materials flowing over flat plates. Analogous extensions to mass transfer are indicated.

A tentative criterion is suggested for characterizing the transition from laminar to turbulent boundary-layer flow of power law fluids.

Relationships combining the effects of a part laminar, part turbulent boundary layer are presented.

Increasing attention is being focused upon the engineering aspects of non-Newtonian flow. Among examples of the importance of non-Newtonian boundary-layer theory may be cited the possibility of reducing frictional drag on bearings and on immersed bodies, such as ships' hulls and submarines, by coating them with a layer of an appropriate pseudoplastic fluid. This paper will consider momentum, heat, and mass transfer relationships in the turbulent boundary-layer flow of power law fluids over a smooth flat plate.

THE TURBULENT BOUNDARY LAYER IN FLOW OF POWER LAW FLUIDS OVER A SMOOTH FLAT PLATE

The boundary-layer equations for steady flow of a constant property fluid over a flat plate at zero incidence are well known to be (*14*, Chap. VII):

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{g_c}{\rho}\frac{\partial \tau_{xy}}{\partial y} \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

These expressions may be integrated to obtain Von Karman's integral momentum equation for the boundary layer (*13*, p. 42):

$$\frac{\tau_o\, g_c}{\rho} = \frac{d}{dx}\int_o^\delta (u_o - u)\, u\, dy \tag{3}$$

An expression for the velocity profile and a second relationship for local shear stress at the plate surface τ_o may be obtained from an equation presented by Dodge and Metzner (*6*) relating friction factor to generalized Reynolds number for fully developed turbulent non-Newtonian flow in tubes:

$$f = \frac{\alpha}{(N_{Re\ gen})^{\beta}};\ 5 \times 10^3 \leq N_{Re\ gen} \leq 10^5 \tag{4}$$

where

$$N_{Re\ gen} = \frac{D^{n'}\, V^{2-n'}\, \rho}{\gamma}$$

α, β = known (graphical) functions of n' (*6*).

Now the power law rheological equation is

$$\tau_{xy} = \frac{K}{g_c}\left(\frac{\partial u}{\partial y}\right)^n \tag{5}$$

and in the special case of power law fluids (*12*)

$$n' = n \tag{6}$$

$$\gamma = \gamma_1 = 8^{n-1} K \left(\frac{3n+1}{4n}\right)^n \tag{7}$$

so that Equation (4) becomes

$$f = \tau_w\, g_c \Big/ \frac{\rho V^2}{2} = \frac{\alpha\, \gamma_1^{\beta}}{D^{\beta n}\, V^{\beta(2-n)}\, \rho^{\beta}}$$

rearranging and replacing D by $2R$

$$\tau_w\, g_c = \left(\frac{\alpha}{2^{\beta n+1}}\right)\rho\, V^{2-\beta(2-n)}\left(\frac{\gamma_1}{\rho}\right)^{\beta} R^{-\beta n} \tag{8}$$

The turbulent velocity profile will be assumed to have the form

$$\frac{u}{u_{max}} = \left(\frac{y}{R}\right)^q \tag{9}$$

where y is the distance from the tube wall and q is an unknown exponent which will be evaluated by a procedure analogous to that used for Newtonian fluids (*10*, pp. 71, 72):

$$V = (\text{constant})\, u_{max} = (\text{constant})\left(\frac{R}{y}\right)^q u$$

By substituting this expression for V in Equation (8) and by solving for u, one obtains

$$u = \left(\frac{\tau_w g_c}{A}\right)^{\frac{1}{2-\beta(2-n)}} R^{\frac{\beta n - q[2-\beta(2-n)]}{2-\beta(2-n)}}\, y^q \tag{10}$$

where

$$A = \left(\frac{\alpha}{2^{\beta n+1}}\right)\rho\left(\frac{\gamma_1}{\rho}\right)^{\beta}(\text{constant})^{2-\beta(2-n)}$$

Near the wall the flow is hardly distinguishable from that in a true boundary layer, so that the velocity distribution in this region should be independent of R (*9*, *10*, pp. 71, 72). This corresponds to an exponent of zero on R in Equation (10), resulting in

$$q = \frac{\beta n}{2 - \beta(2-n)}$$

and substituting in Equation (9)

$$\frac{u}{u_{max}} = \left(\frac{y}{R}\right)^{\frac{\beta n}{2 - \beta(2-n)}} \tag{11}$$

In the case of Newtonian fluids $n = 1.0$ and $\beta = 0.25$, so that Equation (11) reduces to

$$\frac{u}{u_{max}} = \left(\frac{y}{R}\right)^{1/7}$$

This is the well-known Prandtl one-seventh power law, which has been found experimentally to give "a tolerably good approximation" to the velocity profile, not only near the wall but also over the central portion of flow up to N_{Re} of 10^5 (*10*, pp. 71-72). In view of Bogue and Metzner's (*3*) experimental finding that turbulent velocity profiles for power law fluids "are not substantially different from the Newtonian ones when compared on the basis of u/V," Equation (11) may similarly be expected to provide "a tolerably good approximation," not only near the wall but also over the central portion of flow. This expectation is confirmed in Figures 1 to 3, which compare the experimental non-Newtonian velocity profiles of Bogue and Metzner (*3*) with Equation (11) using $V = 0.817\ u_{max}$. [The averages of the points at the lowest y/R values in Figures 2 and 3 were also below the more complex correlating curve used in the original paper (*3*), which noted increased uncertainties in instrument correction and measurement at low y/R with low n.] An alternative approach analogous to that used by Prandtl (*15*) for Newtonian fluids also results in Equation (11). In the case of Newtonian fluids in turbulent flow the velocity distribution for a tube and a flat plate are found to be effectively the same (*14*, pp. 535-539). By assuming this to hold also for power law fluids, one can modify Equation (11) as follows for flat plates at zero incidence to flow:

$$\frac{u}{u_o} = \left(\frac{y}{\delta}\right)^{\frac{\beta n}{2 - \beta(2-n)}} \tag{12}$$

For turbulent flow of power law fluids in tubes Bogue and Metzner (*3*) found experimentally that V/u_{max} is virtually identical with its value for Newtonian materials, at least for $0.445 \leq n \leq 1.0$. For Newtonian Reynolds numbers of about 10^5, V/u_{max} is equal to 0.817 (*13*, p. 75). Equation (8) may, therefore, be adapted to flow over a smooth flat plate at zero incidence in a manner analogous to that used in the Newtonian case (*2*, *10*, pp. 170-171, *14*, pp. 535-539, *15*), namely, by substituting $0.817\ u_{max}$ for V, u_o for u_{max}, δ for R, and τ_o for τ_w, obtaining

$$\tau_o g_c = \Omega\, \rho^{1-\beta} \gamma_1^{\beta} \delta^{-\beta n} u_o^{2-\beta(2-n)} \tag{13}$$

where

$$\Omega = \frac{\alpha(0.817)^{2-\beta(2-n)}}{2^{\beta n+1}} \tag{14}$$

Equation (13) provides the second expression for $\tau_o g_c$ in a turbulent boundary layer.

Equation (12) is solved for u and the result is substituted in Equation (3) to obtain

$$\tau_o g_c = \psi\, \rho\, u_o^2 \frac{d\delta}{dx} \tag{15}$$

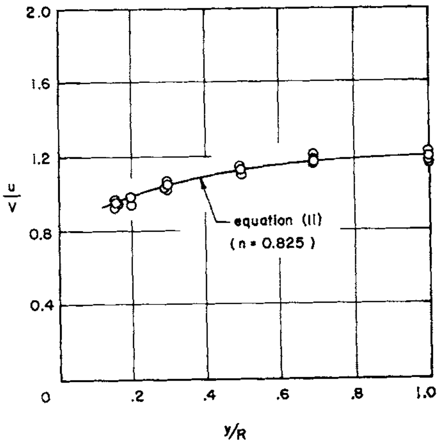

Fig. 1. Comparison between experimental velocity profiles (3) and Equation (11) for moderately non-Newtonian fluids. ○ = Carbopol, n = 0.80 to 0.895, $N_{Re\ gen}$ = 24,960 to 107,500.

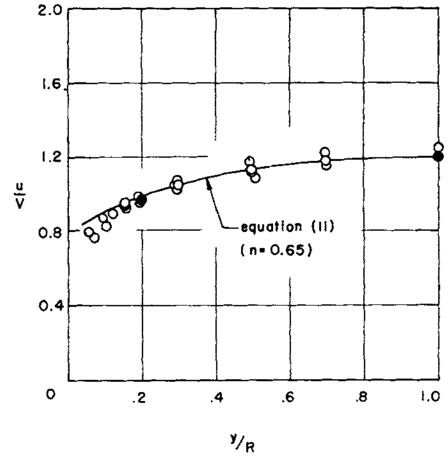

Fig. 2. Comparison between experimental velocity profiles (3) and Equation (11) for intermediate values of n. ○ = Carbopol, n = 0.59 to 0.70, $N_{Re\ gen}$ = 6,100 to 23,240. ● = Carbopol, n = 0.6 to 0.8, $N_{Re\ gen}$ = 46,500 to 85,000.

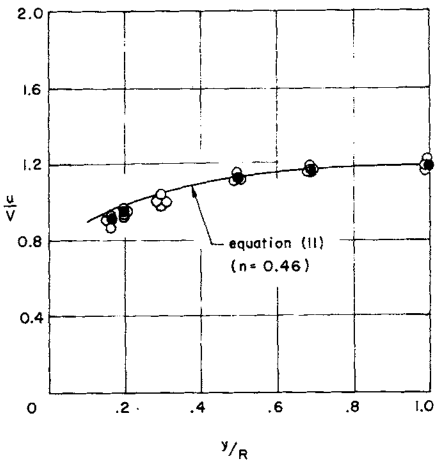

Fig. 3. Comparison between experimental velocity profiles (3) and Equation (11) for highly non-Newtonian fluids (low n). ● = Carbopol, n = 0.445, $N_{Re\ gen}$ = 8,330. ○ = clay suspension, n = 0.46 to 0.53, $N_{Re\ gen}$ = 7,880 to 19,500.

where

$$\psi = \frac{2-\beta(2-n)}{2(1-\beta+\beta n)} - \frac{2-\beta(2-n)}{2-2\beta+3\beta n} \quad (16)$$

The elimination of $\tau_o g_c$ from Equations (13) and (15) gives

$$\int_o^\delta \delta^{\beta n}\, d\delta = \frac{\Omega}{\psi}\left(\frac{\gamma_1}{\rho}\right)^\beta u_o^{-\beta(2-n)} \int_o^x dx$$

It must be noted that the lower limit of zero on the two integral signs implies that the turbulent boundary layer extends over the entire plate from the leading edge. It has been found (5) in the case of Newtonian fluids that this may, in fact, be true if the leading edge of the plate possesses some irregularity which disturbs the flow. In other cases the assumption introduces an error which is small except when the length of the plate is only a little longer than the critical value at which the laminar boundary layer ends.

Integrating

$$\delta = \left[\frac{(\beta n+1)\Omega}{\psi}\left(\frac{\gamma_1}{\rho}\right)^\beta u_o^{-\beta(2-n)}\right]^{\frac{1}{\beta n+1}} x^{\frac{1}{\beta n+1}} \quad (17)$$

This is the thickness of the turbulent boundary layer at a distance x from the leading edge of the plate.

Newtonian boundary-layer relationships for δ, $\tau_o g_c$, total drag force, etc., are customarily expressed in terms of a Reynolds number containing x as the characteristic length dimension. Accordingly, by recalling the definition of γ_1 in Equation (7), we obtain the following two rearranged forms of Equation (17):

$$\delta = \left\{\frac{(\beta n+1)\Omega}{\psi}\left[8^{n-1}\left(\frac{3n+1}{4n}\right)^n\right]^\beta (N^{\circ}_{Rex})^{-\beta}\right\}^{\frac{1}{\beta n+1}} x \quad (18)$$

and

$$\delta = \left[\frac{(\beta n+1)\Omega}{\psi}(N_{ReBx})^{-\beta}\right]^{\frac{1}{\beta n+1}} x \quad (19)$$

where

$$N_{Rex} = \frac{x^n u_o^{2-n}\rho}{K} \quad (20)$$

$$N_{ReBx} = \frac{x^n u_o^{2-n}\rho}{\gamma_1} \quad (21)$$

The transition between laminar and turbulent boundary layer flow of power law materials will later be tentatively characterized in terms of these two forms of Reynolds number.

Differentiating Equation (17) with respect to x and substituting in Equation (15) leads to

$$\tau_o g_c = \frac{\psi\rho u_o^2}{\beta n+1}\left\{\frac{(\beta n+1)\Omega}{\psi}\left[8^{n-1}\left(\frac{3n+1}{4n}\right)^n\right]^\beta (N^{\circ}_{Rex})^{-\beta}\right\}^{\frac{1}{\beta n+1}} \quad (22)$$

This is the local or point value of surface shear stress at distance x from the leading edge of the plate.

The total frictional drag force on one side of the plate of length L and width w is

$$F_{df}\, g_c = w\int_o^L \tau_o g_c\, dx \quad (23)$$

substituting Equation (22) for $\tau_o g_c$

$$F_{df}\, g_c = w\, L\, \psi\, \rho\, u_o^2\left\{\frac{(\beta n+1)\Omega}{\psi}\left[8^{n-1}\left(\frac{3n+1}{4n}\right)^n\right]^\beta (N^{\circ}_{ReL})^{-\beta}\right\}^{\frac{1}{\beta n+1}} \quad (24)$$

In the special case of Newtonian fluids $n = 1$, $K = \mu$, $\gamma_1 = \mu$, $\alpha = 0.0791$, and $\beta = 0.25$. Appropriate substitution into Equations (14) and (16) shows that $\Omega = 0.0232$ and $\psi = 7/72$. Insertion of these values into Equation (24) results in

$$F_{df}\, g_c = 0.036\,\rho\, u_o^2\, w\, L\left(\frac{\mu}{L\, u_o\, \rho}\right)^{1/5} \quad (25)$$

which is the well-known expression for Newtonian materials (*15*).

A *different* approach to drag prediction has been made by Granville (*8*), who used logarithmic forms of velocity distribution. His resulting expressions are substantially more complex, as might be expected from the corresponding Newtonian case (*14*, pp. 539-541), and were not extended to heat or mass transfer considerations.

The equations so far derived have involved the assumption that Equation (12) applies throughout the boundary layer, namely, for $0 \leq y \leq \delta$. The equivalent assumption for Newtonian fluids regarding the approximate validity of the Prandtl one-seventh power law over this range is customary and the results show good agreement with experiment, except at very high Reynolds numbers. Actually, however, a thin laminar sublayer exists between the solid surface and the turbulent boundary layer. It is reasonable to expect that the velocity profile is approximately linear within this thin laminar sublayer, so that the velocity gradient and the shear stress are constants in this region. The added approximation resulting from the use of Equation (12) over the range $0 \leq y \leq \delta$, therefore, depends on the thickness of the laminar sublayer, which may be expected to be thinner than in the Newtonian case for n below unity and thicker for n above unity. Confirmation of this speculation is provided in the next section, where an estimate is made of the thickness of the laminar sublayer as a function of distance along the plate. The buffer zone will be neglected in this treatment.

The thickness of the laminar sublayer is also required when defining whether or not a plate is "smooth" in the fluid dynamics sense. Thus a surface is smooth when surface protuberances are wholly within the laminar sublayer.

THE LAMINAR SUBLAYER

At a distance x from the leading edge of the plate, the thickness of the turbulent boundary layer is δ; the thickness of the corresponding laminar sublayer at this point will be denoted by δ_L. Because the laminar sublayer is so thin the velocity gradient will be effectively constant for y less than δ_L, so that

$$\tau_o g_c = K\left(\frac{\partial u}{\partial y}\right)^n\bigg|_{y=0} = K\left(\frac{u}{y}\right)^n \text{for } y \leq \delta_L$$

and from Equation (13)

$$\tau_o g_c = \Omega\,\rho^{1-\beta}\,\gamma_1^\beta\,\delta^{-\beta n}\,u_o^{2-\beta(2-n)} = K\left(\frac{u}{y}\right)^n$$

$$= \Omega \rho u_o^2 (N_{Re\,B\,\delta})^{-\beta} = K\left(\frac{u}{y}\right)^n$$

where

$$N_{Re\,B\,\delta} = \frac{\delta^n u_o^{2-n} \rho}{\gamma_1}$$

let

$$u = u_L \quad \text{at} \quad y = \delta_L$$

then

$$u_L^n = \frac{\Omega \rho u_o^2}{K} \frac{\lambda_1}{\delta^n u_o^{2-n} \rho} (N_{Re\,B\,\delta})^{1-\beta} \delta_L^n$$

$$\frac{\delta_L}{\delta} = \left[\frac{K}{\Omega \gamma_1}\right]^{\frac{1}{n}} \frac{u_L}{u_o} (N_{Re\,B\,\delta})^{\frac{\beta - 1}{n}} \tag{26}$$

Now the velocity at the inner edge of the turbulent section of the boundary layer (that is, at $y = \delta_L$) must be that given by Equation (12), from which

$$\frac{\delta_L}{\delta} = \left(\frac{u_L}{u_o}\right)^{\frac{2-\beta(2-n)}{\beta n}} \tag{27}$$

From Equations (26) and (27)

$$\frac{u_L}{u_o} = \left[\frac{K}{\Omega \gamma_1}\right]^{\frac{\beta}{2(1-\beta)}} (N_{Re\,B\,\delta})^{-\beta/2} \tag{28}$$

It is desirable to convert Equation (28) from terms of $N_{Re\,B\,\delta}$ to terms of $N_{Re\,Bx}$. For this purpose Equation (17) will be substituted for δ in $N_{Re\,B\,\delta}$ to obtain

$$\frac{u_L}{u_o} = \left[\frac{K}{\Omega \gamma_1}\right]^{\frac{\beta}{2(1-\beta)}} \left\{\left[\frac{(\beta n + 1)\Omega}{\psi}\right]^{\frac{n}{\beta n + 1}} (N_{Re\,Bx})^{\frac{1}{\beta n + 1}}\right\}^{-\beta/2} \tag{29}$$

Equation (29) is substituted for u_L/u_o in Equation (27) to give

$$\frac{\delta_L}{\delta} = \left[\frac{K}{\Omega \gamma_1}\right]^{\frac{2-2\beta+\beta n}{2n-2\beta n}} \left\{\left[\frac{(\beta n + 1)\Omega}{\psi}\right]^{\frac{n}{\beta n + 1}} (N_{Re\,Bx})^{\frac{1}{\beta n + 1}}\right\}^{\frac{2\beta - 2 - \beta n}{2n}} \tag{30}$$

Equation (19) is next solved for δ/x and the result multiplied by Equation (30) to give the following expression after some simplification:

$$\frac{\delta_L}{x} = \left[\frac{K}{\Omega \gamma_1}\right]^{\frac{2-2\beta+\beta n}{2n-2\beta n}} \left[\frac{(\beta n + 1)\Omega}{\psi}\right]^{\frac{2\beta - \beta n}{2(\beta n + 1)}} (N_{Re\,Bx})^{\frac{2\beta - 2 - 3\beta n}{2n(\beta n + 1)}} \tag{31}$$

For Newtonian materials

$n = 1, K = \mu, \gamma_1 = \mu, \alpha = 0.0791,$

$\beta = 0.25, \Omega = 0.0232$ and $\psi = 7/72;$

also $N_{Re\,Bx} = N_{Rex}$. Insertion of these values into Equation (31) gives

$$\frac{\delta_L}{x} = 71.5\,(N_{Re\,x})^{-0.9} \tag{32}$$

The terms γ_1, Ω, and ψ are each complex functions of n, defined by Equations (7), (14), and (16), respectively. Furthermore, α and β vary with n (*6*). The manner in which the thickness of the laminar sublayer varies with n is, therefore, not perhaps readily apparent from inspection of Equations (30) and (31). Numerical substitutions, however, confirm that δ_L/δ and δ_L/x decrease as n decreases at constant x.

TRANSITION FROM LAMINAR TO TURBULENT BOUNDARY-LAYER FLOW

Several variables determine the transition from laminar to turbulent flow in the boundary layer on a flat plate. These include the degree of turbulence in the free stream, the amount of roughness of the plate surface, heat transfer across the plate surface, and the value of some form of Reynolds number which characterizes the flow. In the case of Newtonian fluids transition occurs within the following range of Reynolds number

$$3 \times 10^5 < x_c u_o \rho/\mu < 3 \times 10^6$$

where x_c is the critical distance along the surface from the leading edge.

No such criterion has been established presently for boundary-layer flow of non-Newtonian materials. In the case of power law fluids in a boundary layer, however, a tentative criterion for flow transition will be suggested as follows.

For the *laminar* boundary layer flow of power law fluids over a flat plate at zero incidence, the following expression holds (*1*)

$$\frac{\tau_o g_c}{\rho u_o^2} = c(n)\,(N^{\circ}_{Re\,x})^{-\frac{1}{n+1}} \tag{33}$$

where $c(n)$ is dependent on n. Exact values for $c(n)$ have been calculated for various n by Acrivos, Shah, and Petersen (*1*). For Newtonian materials, $n = 1.0$ and $c(n) = 0.33206$. An effective viscosity at x may now be written as μ_{ex}, where μ_{ex} is that value of effective viscosity which will cause the Newtonian expression to fit the power law case. Accordingly

$$\frac{\tau_o g_c}{\rho u_o^2} = c(n)\,(N^{\circ}_{Re\,x})^{-\frac{1}{n+1}} = 0.33206 \left(\frac{x u_o \rho}{\mu_{ex}}\right)^{-\frac{1}{2}}$$

or

$$\frac{x u_o \rho}{\mu_{ex}} = \left(\frac{0.33206}{c(n)}\right)^2 \left(\frac{x^n u_o^{2-n} \rho}{K}\right)^{\frac{2}{n+1}}$$

$$= \left(\frac{0.33206}{c(n)}\right)^2 \left(\frac{\gamma_1}{K} N_{Re\,Bx}\right)^{\frac{2}{n+1}}$$

The tentative criterion for transition from laminar to turbulent boundary layer flow of power law fluids is, therefore,

$$3 \times 10^5 < \left(\frac{0.33206}{c(n)}\right)^2 \left(\frac{x_c^n u_o^{2-n} \rho}{K}\right)^{\frac{2}{n+1}} < 3 \times 10^6 \tag{34}$$

Clearly there is a need for experimental verification of this criterion. In particular, analogy with tube flow (*6*) suggests that the transition criterion increases slowly as n decreases.

The total frictional drag force on one side of a plate for which L exceeds x_c is estimated as

$$(F_{df})_{\text{total}} = (F_{df})_1 - (F_{df})_2 + (F_{df})_3 \qquad (35)$$

where $(F_{df})_{1 \text{ and } 2}$ are calculated from Equation (24) with $x = L$ and x_c, respectively. $(F_{df})_3$ is evaluated from Equations (23) and (33) for $x = x_c$, obtaining x_c from Equation (34).

TURBULENT THERMAL BOUNDARY LAYERS

Consider a power law fluid flowing over an isothermal flat plate whose temperature is T_s. The fluid at points remote from the surface has a different temperature T_o. Fluid properties will be regarded as effective constants if $T_s - T_o$ is small; heat transfer begins at $x = 0$ in all cases.

Equation (22) gives the local or point value of surface shear stress (that is, of x momentum flux in the y direction) at distance x from the leading edge of the plate under turbulent conditions. The expression may be reduced to the Newtonian case by substituting $n = 1.0$, $K = \gamma_1 = \mu$, $\alpha = 0.0791$, $\beta = 0.25$, $\Omega = 0.0232$, and $\psi = 7/72$, giving

$$\tau_o g_c = 0.0296\, \rho\, u_o^2 \left(\frac{x\, u_o\, \rho}{\mu} \right)^{-1/5} \qquad (36)$$

An effective viscosity at x may now be written as μ_{ex}, and is defined as that viscosity which makes the Newtonian Equation (36) fit the turbulent power law conditions described by Equation (22). The term μ_{ex} is next substituted for μ in Equation (36), the result set equal to Equation (22) and solved for μ_{ex} to obtain

$$\mu_{ex} = u_o\, \rho \left[\frac{\psi}{0.0296\, (\beta n + 1)} \right]^5 \left[\frac{(\beta n + 1)\Omega}{\psi} \left(\frac{\gamma_1}{\rho\, u_o^{2-n}} \right)^\beta \right]^{\frac{5}{\beta n + 1}} x^{\frac{1 - 4\beta n}{\beta n + 1}} \qquad (37)$$

Colburn's analogy between heat and momentum transfer may be readily applied to turbulent flow over flat plates as follows (*11*, *13*, pp. 197-200).

$$\frac{h_x}{c_p\, \rho\, u_o} \left(\frac{c_p\, \mu_{ex}}{k} \right)^{2/3} = j'_H = \frac{C'_{Df}}{2} = \frac{\tau_o\, g_c}{\rho\, u_o^2} = 0.0296 \left(\frac{\mu_{ex}}{x\, u_o\, \rho} \right)^{1/5} \qquad (38)$$

where j'_H and C'_{Df} are local values of j_H and frictional drag coefficient, respectively. Equation (37) is substituted for μ_{ex} in Equation (38) and the result solved for the local value of the heat transfer coefficient h_x as

$$h_x = 0.0296\, (k^2\, c_p\, \rho\, u_o)^{1/3} \left[\frac{0.0296\, (\beta n + 1)}{\psi} \right]^{7/3} \left[\frac{(\beta n + 1)\Omega}{\psi} \left(\frac{\gamma_1}{\rho\, u_o^{2-n}} \right)^\beta \right]^{\frac{-7}{3(\beta n + 1)}} x^{\frac{5\beta n - 2}{3(\beta n + 1)}} \qquad (39)$$

In certain cases, as, for example, when the leading edge of the plate is rough, the boundary layer may be turbulent over the whole plate (*5*). Under these conditions, the average coefficient over the entire plate may be obtained as

$$h_m = \frac{1}{L} \int_o^L h_x \,[\text{Equation (39)}]\, dx$$

from which

$$h_{m\,\text{turb.}} = 0.0888 \left(\frac{\beta n + 1}{8\, \beta n + 1} \right) (k^2\, c_p\, \rho\, u_o)^{1/3} \left[\frac{0.0296(\beta n + 1)}{\psi} \right]^{7/3} \left[\frac{(\beta n + 1)\Omega}{\psi} \left(\frac{\gamma_1}{\rho\, u_o^{2-n}} \right)^\beta \right]^{\frac{-7}{3(\beta n + 1)}} L^{\frac{5\beta n - 2}{3(\beta n + 1)}} \qquad (40)$$

Equation (40) will reduce to the known Newtonian form (*11*) as a special case.

In the case of a hydrodynamically smooth plate, however, the presence of the laminar boundary layer over the range $0 \leq x \leq x_c$ would cause Equation (40) to give an overestimate of h_m which would be in significant error unless $L >> x_c$. For this situation expressions for local h in the region $x < x_c$ (that is, the laminar boundary-layer region) are needed. Such expressions have been provided by Acrivos, Shah, and Petersen (*1*) as follows:

$$h_x = \frac{k}{(0.893)x} \left[\frac{c(n)^{1/n}}{9} \frac{1 + 2n}{2 + 2n} \right]^{1/3} (N^{\circ}_{Re\,L})^{\frac{1}{1+n}} (N^{\circ}_{Pr})^{1/3} \left(\frac{x}{L} \right)^{\frac{1 + 2n}{3(1 + n)}} \qquad (41)$$

$$h_x = \frac{k}{x\sqrt{\pi}} \left(\frac{x}{L} \right)^{1/2} (N^{\circ}_{Re\,L})^{\frac{1}{1+n}} (N^{\circ}_{Pr})^{1/2} \qquad (42)$$

$$\left(\frac{x^*}{L} \right)^{\frac{n - 1}{6(1 + n)}} = 1.3157\, (N^{\circ}_{Pr})^{1/6} \left[\frac{2 + 2n}{1 + 2n} \left(\frac{1}{c(n)} \right)^{1/n} \right]^{1/3} \qquad (43)$$

When $n < 1.0$, h_x is given by Equation (42) for $0 \leq x \leq x^*$ and by Equation (41) for $x > x^*$.

When $n > 1.0$, h_x is calculated from Equation (41) for $0 \leq x \leq x^*$ and from Equation (42) for $x > x^*$.

The variation in x^* with increasing Prandtl number is such that Equation (41) applies over by far the greater portion of the plate surface when N_{Pr} is large.

For a hydrodynamically smooth plate, then, assuming that the turbulent boundary layer extrapolates to $\delta = 0$ at $x = 0$, one may estimate the mean coefficient over the entire plate by using an extension of the expression given by Rohsenow and Choi (*13*, pp. 197-200), illustrated here for $n < 1.0$:

$$h_m = \frac{1}{L} \left[\int_o^{x^*} h_x \,[\text{Equation (42)}]\, dx + \int_{x^*}^{x_c} h_x \,[\text{Equation (41)}]\, dx + \int_{x_c}^{L} h_x \,[\text{Equation (39)}]\, dx \right] \qquad (44)$$

Insertion of Equations (39), (41), and (42) in this expression results in

$$h_m = \frac{k}{L} (N^{\circ}_{Re\,L})^{\frac{1}{1+n}} (N^{\circ}_{Pr})^{1/3} \left\{ \frac{2}{\sqrt{\pi}} \left(\frac{x^*}{L} \right)^{1/2} (N^{\circ}_{Pr})^{1/6} + \right.$$

$$\frac{3(1+n)}{0.893(1+2n)}\left[\frac{c(n)^{1/n}}{9}\,\frac{1+2n}{2+2n}\right]^{1/3}$$

$$\left[\left(\frac{x_c}{L}\right)^{\frac{1+2n}{3(1+n)}}-\left(\frac{x^*}{L}\right)^{\frac{1+2n}{3(1+n)}}\right]\Bigg\}+$$

$$0.0888\left(\frac{k}{L}\right)\left(\frac{\beta n+1}{8\beta n+1}\right)\left(\frac{c_p\,\rho\,u_o}{k}\right)^{1/3}$$

$$\left[\frac{0.0296(\beta n+1)}{\psi}\right]^{7/8}\left[\frac{(\beta n+1)\Omega}{\psi}\left(\frac{\gamma_1}{\rho\,u_o^{2-n}}\right)^{\beta}\right]^{\frac{-7}{3(\beta n+1)}}$$

$$\times\left\{L^{\frac{8\beta n+1}{3(\beta n+1)}}-x_c^{\frac{8\beta n+1}{3(\beta n+1)}}\right\} \qquad (45)$$

The value of x^* is obtained from Equation (43) and x_c from Equation (34). An expression analogous to Equation (45) could, of course, be obtained for $n > 1.0$, noting specifications (*a*) and (*b*) beneath Equation (43). It may at first be thought that, since Equations (41) and (42) are for laminar flow, the length term in these equations should be x_c when used in Equation (44). Inspection shows, however, that Equations (41) to (43) are actually independent of L, so that use of L where $L > x_c$ is not significant. Equation (45) is for use with an isothermal plate and the temperature difference $T_s - T_o$.

The limits stated on Equation (4) determine the range of validity of these relationships, which may be estimated in a manner similar to that given by Schlichting (*14*, p. 537) as follows. From Equations (4) and (7)

$$\frac{D^n V^{2-n}\rho}{\gamma_1}=\frac{(2R)^n\,(0.817\,u_{max})^{2-n}\,\rho}{\gamma_1}\leq 10^5$$

so that for the plate (*14*, p. 537)

$$(0.817)^2\left(\frac{2}{0.817}\right)^n N_{Re\,B\delta}\leq 10^5$$

and substituting for δ from Equation (19)

$$N_{Re\,Bx}\leq\left[\frac{(1.5)\,10^5}{(2.445)^n}\right]^{\beta n+1}\left[\frac{(\beta n+1)\,\Omega}{\psi}\right]^{-n} \qquad (46)$$

For Newtonian fluids this reduces to $N_{Rex} \leq 3.24 \times 10^6$, which is virtually identical with the upper limit given by Goldstein (*7*) on Equation (25) when the boundary layer is turbulent from the leading edge of the plate. Schlichting (*14*, p. 537), on the other hand, used the same experimental data to infer an upper limit of 10^7, so that Equation (46) probably gives a conservative estimate for the upper limit of validity of these expressions. For still higher values of N_{ReBx}, it would be desirable to reevaluate α and β at the appropriate n from Dodge and Metzner's (*6*) recommended extrapolation of their Figure 12; V/u_{max} in Equatiton (14) would also increase slightly (*13*, p. 75).

Finally, it is evident that expressions entirely analogous to Equations (39) and (40) can be developed similarly for the local and mean coefficients of *mass* transfer between a power law fluid in turbulent boundary-layer flow and a flat plate at zero incidence. This would be done after replacing j_H' in Equation (38) with j_D', the local value of the *mass* transfer j factor (*13*, pp. 413-416). Application would involve prediction of molecular diffusivities in non-Newtonian systems by published methods (*4*).

NOTATION

$c(n)$ = constant, dependent on n
c_p = specific heat, B.t.u./(lb.$_m$) (°R.)
D = tube diameter, ft.
f = Fanning friction factor, dimensionless
F = force, lb.$_f$
F_{df} = frictional drag force, lb.$_f$
g_c = conversion factor, 32.174 (lb.$_m$) (ft.)/(lb.$_f$) (sec.^{2})
h = coefficient of heat transfer, B.t.u./(hr.) (sq.ft.) (°R.)
h_m, h_x = mean and local values of heat transfer coefficient, B.t.u./(hr.) (sq.ft.) (°R.)
j_H', j_D' = local values of heat and mass transfer j factors, dimensionless
k = thermal conductivity, B.t.u./(hr.) (sq.ft.) (°R.)/ft.
K = fluid consistency index, lb.$_m$ (sec.$^{n-2}$) (ft.$^{-1}$)
L = length of plate in the direction of flow, ft.
n = flow behavior index, dimensionless
N°_{Pr} = $c_P u_o \rho L/k(N^\circ_{ReL})^{2/(1+n)}$
N_{ReBx} = Reynolds number defined as $x^n u_o^{2-n}\rho/\gamma_1$
$N_{ReB\delta}$ = Reynolds number defined as $\delta^n\, u_o^{2-n}\rho/\gamma_1$
$\overset{\circ}{N}_{ReL}$ = Reynolds number defined as $L^n u_o^{2-n}\rho/K$
N_{Rex} = Reynolds number defined as $x u_o \rho/\mu$
$\overset{\circ}{N}_{Rex}$ = Reynolds number defined as $x^n u_o^{2-n}\rho/K$
q = exponent
R = tube radius, ft.
u = linear velocity in the boundary layer (that is, for $y < \delta$) and parallel to the solid surface, ft./sec.
u_L = linear velocity at $y = \delta_L$, ft./sec.
u_o = linear velocity of the undisturbed stream, ft./sec.
V = mean linear velocity, ft./sec.
v = velocity component in y direction, ft./sec.
w = plate width normal to flow, ft.
x = distance along surface from leading edge and in the direction of flow, ft.
x_c = critical value of x at which laminar flow ends in the boundary layer, ft.
x^* = distance defined by Equation (43), ft.
y = distance normal to surface, ft.

Greek Letters

α, β = dimensionless functions of n
γ_1 = defined by Equation (7)
δ = boundary-layer thickness at a given value of x, ft.
δ_L = thickness of laminar sublayer at a given x, ft.
μ = viscosity, lb.$_m$/(ft.) (sec.)
ρ = density, lb.$_m$/cu.ft.
τ_o = shear stress at the solid surface (that is, at y = 0) lb.$_f$/sq.ft.
τ_w = shear stress at a tube wall, lb.$_f$/sq.ft.
ψ, Ω = defined by Equations (16) and (14), respectively

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Manuscript received February 9, 1965; revision received August 13, 1965; paper accepted August 18, 1965.

Characterization and Gain Identification of Time Varying Flow Processes

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A method for continuously estimating the gain of a flow process by sinusoidal perturbation is presented. The resulting output perturbation is correlated with a second sinusoid to generate periodically an estimate of the process gain. A method of implementing such an identifier on a small analog computer is described.

The experimental testing of this identifier computer with both a real process (a pH regulating system) and with an analog computer simulation of the process is described. The results of identification tests with a nonstationary system are presented. From these results it is concluded that the identifier estimates the process gain satisfactorily, introducing a delay (equal to one-half the period of identification) and making an effective sampling or clamping of the gain estimate (over each period of identification).

A large class of problems in the process field involves the control of complex, multivariable dynamic systems. From the point of view of control theory, the most desirable approach is to consider all the process variables; that is, the process outputs (state variables) are measured and the process inputs (manipulated variables) are regulated to control the system in some desired fashion. In practice this is seldom done; perhaps the major reasons for not using such an approach are that it is usually difficult to measure certain types of state variables (for example, chemical concentrations), and that the detailed knowledge of the process dynamics often is lacking.

More often, an easily measured output (such as temperature) and an easily manipulated input (a stream flow rate) are chosen. The approximate process dynamics relating these two variables are obtained (often this step involves experimental testing) and a conventional control system is installed. Considering the system as a single input/single output process* has many obvious advantages, notably the relative ease of control system design. The disadvantage of this approach is that the effects on the process dynamics of the disregarded process variables may be important. Hence a controller designed for one set of process conditions may not be satisfactory if the conditions change.

An alternative approach which may be used with flow systems is to retain the single input-single output formulation of the process, but to take into account the possibility of time variation in process characteristics which may result from variation in the unmeasured state variables. For example, a stirred tank chemical reactor may be controlled by measuring the temperature of the reacting mixture and by adjusting cooling water flow rate to maintain this quantity fixed. However, the process dynamics are known to be functions of the contents of the reactor and hence will vary with the concentrations of reactant, product, and catalyst.

The problem, then, becomes one of measuring process dynamics continuously and adjusting the controller parameters to compensate for variation in process dynamics. Such a control approach is generally termed adaptive; the controller adapts itself to maintain satisfactory control in spite of a time varying process.

This paper treats a subclass of such systems: gain varying flow processes. Thus, approximate mathematical analysis of a chemical reactor, such as that mentioned above, can be effected by considering the system equations linearized about the operating point. This may indicate the possibility of time variation in the process gain. The mathematical analysis in the next section will show the pH regulating system to be such a process.

A particular method of determining the process gain will be used; that is, the control system input will be perturbed sinusoidally. The resulting (approximately) sinusoidal perturbation in the measured system output will furnish a periodic estimate of the process gain. The necessary theory will be developed and experimental verification furnished. A companion paper will consider the problem of constructing a control system to use such information.

* Single input in the sense that there is only one manipulated input.

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